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Timelike Killing spinors in seven dimensions

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ABSTRACT: We employ the G-structure formalism to study supersymmetric solutions of minimal and $SU(2)$ gauged supergravities in seven dimensions admitting Killing spinors with associated timelike Killing vector. The most general such Killing spinor defines an $SU(3)$ structure. We deduce necessary and sufficient conditions for the existence of a timelike Killing spinor on the bosonic fields of the theories, and find that such configurations generically preserve one out of sixteen supersymmetries. Using our general supersymmetric ansatz we obtain numerous new solutions, including squashed or deformed AdS solutions of the gauged theory, and a large class of Gödel-like solutions with closed timelike curves.

KEYWORDS: seven-dimensional supergravities, G-structures.

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1. Introduction

In recent years, gauged supergravities in various dimensions have been recognised as powerful technical tools for the construction of interesting supersymmetric backgrounds of string/ M theory. A particularly fruitful application has been to the generalised AdS/CFT correspondence. Following the celebrated work of Maldacena and Nunez, [1], [2], an extensive literature has developed on branes wrapping various supersymmetric cycles, and their associated field theories; a review and examples of applications of the supergravity aspects are to be found in [3]-[7]. On the gravity side, the near-horizon geometries are most easily constructed in some lower dimensional gauged supergravity, before lifting to $d = 10$ or 11. In view of this and other applications, it is clearly important to have some systematic understanding of supersymmetric solutions in gauged supergravities.

More generally, the ability to map out all supersymmetric backgrounds in any conceivable (supergravity) limit of string/ M theory is of much value. In fact, the framework which allows one to achieve precisely this has been identified, and is provided by the notion of a G-structure. Already, much work has been done in various contexts, [8]-[21], and a classification of all supersymmetric solutions of $d = 11$ supergravity has been given [22], [23].

Geometrically, G-structures provide information about supersymmetric field configurations. Typically one finds that fluxes in a given theory are tightly constrained (and sometimes entirely fixed) by the torsion classes of the structure. However, it must be stated that the approach is not without its drawbacks. Generally speaking, the classification becomes progressively more implicit as the dimensionality of spacetime and/ or the number of supercharges increases. The point is that while the existence of a G-structure implies and is implied by the vanishing of the supersymmetry variations of the fermions in the theory, including the Killing spinor equation, it is not equivalent to having a solution of the field equations and Bianchi identities. When a Killing spinor exists, generically some but not all of these are satisfied identically. The remainder must be imposed as additional constraints on the bosonic fields, over and above those implied by the G-structure. Typically, they take the form of differential constraints on the torsion classes of the structure, and these are often hard to solve. Nevertheless, the generality of the technique, and its constructive nature in providing a clear set of prescriptions for constructing explicit solutions, makes it very powerful.

In this paper we will apply the G-structure formalism to minimal and $SU(2)$ gauged supergravity in seven dimensions. We will focus on configurations admitting a timelike Killing spinor - that is, a Killing spinor whose associated Killing vector is timelike. The null case will be the subject of a future work. We find that a timelike Killing spinor is equivalent to an $SU(3)$ structure, and determine the necessary and sufficient conditions on the bosonic fields for its existence.

One of the more surprising and unsettling features of supergravities revealed by G-structures is the extent to which the class of (particularly timelike) supersymmetric solutions is infested by spacetimes with closed timelike curves. We find many new examples here, in the gauged theory as bundles over negative scalar curvature Kähler threefolds, which lift to $d = 10, 11$. We show how various $AdS_{7,5,3}$ solutions of the theory, for which the closed timelike curves may be eliminated, arise as special cases of this much broader class of solutions, and also how the AdS factors may be squashed or deformed by the addition of suitable fluxes.

The plan of the paper is as follows. In section two we describe the theories we study. In section three we give a summary of the necessary and sufficient conditions for a bosonic field configuration in the theories to admit a timelike Killing spinor. The derivation of these conditions is quite technical, and is relegated to the appendices. In sections four and five we obtain explicit supersymmetric solutions of the minimal and gauged theories respectively. Section six concludes. In appendix A we give our conventions, and miscellaneous useful material. In appendix B we compute the various bilinears that may be constructed from a Killing spinor, and show how they define an $SU(3)$ structure in seven dimensions. In appendix C we derive the necessary conditions for a bosonic configuration to admit a timelike Killing spinor, and show that these are also sufficient. In appendix D we discuss the intrinsic torsion and contorsion of an $SU(3)$ structure in six Riemannian dimensions. In appendix F we give the integrability conditions for the theories.

2. The theory and supersymmetry variations

The lagrangian for minimal seven dimensional ungauged supergravity was first written down in [25]. The $SU(2)$ gauged version was also written down in [25] but with numerical typos, which were corrected in [26]. In [25] Euclidean signature is used.

2.1 The minimal theory

The bosonic lagrangian density for the minimal theory in our conventions is

$$\mathcal{L} = \frac{1}{2}eR - \frac{1}{24}e(G_{\mu\nu\rho\tau})^2 - \frac{1}{2}eF_{\mu\nu}{}^a{}_b F^{\mu\nu}{}^b{}_a - \frac{5}{2}e(\partial_\mu\phi)^2$$
$$- \frac{1}{24}ee^{-\phi}G_{\mu\nu\rho\tau}F_{\kappa\lambda}{}^a{}_b A_\chi{}^b{}_a \epsilon^{\mu\nu\rho\tau\kappa\lambda\chi}. \tag{2.1}$$

Compared to [25] we use the same conventions for the Riemann tensor but Hawking and Ellis conventions for the Ricci tensor and scalar. We have also rescaled $\phi \to \sqrt{5}\,\phi$ and the forms by $F \to \sqrt{2}\,e^\phi F$, $G \to \sqrt{2}\,e^{-2\phi}G$, $A_{(1),(3)} \to \sqrt{2}A_{(1),(3)}$. The

supersymmetry variations of the fermions are given by

$$\delta\lambda^a = \frac{\sqrt{5}}{2}\Gamma^\mu D_\mu\phi\,\epsilon^a + \frac{i}{2\sqrt{5}}\Gamma^{\mu\nu}F_{\mu\nu}{}^a{}_b\,\epsilon^b + \frac{1}{24\sqrt{5}}\Gamma^{\mu\nu\rho\tau}G_{\mu\nu\rho\tau}\,\epsilon^a, \tag{2.2}$$

$$\delta\psi_\mu^a = D_\mu\,\epsilon^a - \frac{i}{10}(\Gamma_\mu{}^{\nu\rho} - 8\delta_\mu^\nu\Gamma^\rho)F_{\nu\rho}{}^a{}_b\,\epsilon^b + \frac{1}{80}(\Gamma_\mu{}^{\alpha\beta\gamma\delta} - \frac{8}{3}\delta_\mu^\alpha\Gamma^{\beta\gamma\delta})G_{\alpha\beta\gamma\delta}\,\epsilon^a\,, \tag{2.3}$$

and the parameter ϵ^a is a symplectic-Majorana spinor, whose properties are summarized in appendix A.

Let us introduce the following notation. Let A_p, B_q be p- and q-forms respectively. Then

$$A_\lrcorner B_{a_1\ldots a_{q-p}} = \frac{1}{p!}A^{b_1\ldots b_p}B_{b_1\ldots b_p a_1\ldots a_{q-p}}. \tag{2.4}$$

The equations of motion and Bianchi identities are

$$d(e^{-2\phi}G) = 0, \tag{2.5}$$

$$d(e^\phi F^A) = 0, \tag{2.6}$$

$$P = 5\nabla^2\phi - 4G_\lrcorner G + F^A\lrcorner F^A = 0, \tag{2.7}$$

$$Q = \star(e^{-2\phi}d\star(e^{2\phi}G) - \frac{1}{2}F^A\wedge F^A) = 0, \tag{2.8}$$

$$R^A = \star(e^\phi d\star(e^{-\phi}F^A) - 2F^A\wedge G) = 0, \tag{2.9}$$

$$E_{\mu\nu} = R_{\mu\nu} - \frac{1}{3}\Big(G_{\mu\alpha\beta\gamma}G_\nu{}^{\alpha\beta\gamma} - \frac{1}{10}g_{\mu\nu}G_{\alpha\beta\gamma\delta}G^{\alpha\beta\gamma\delta}\Big) - 5\partial_\mu\phi\partial_\nu\phi$$
$$- \Big(F_{\mu\alpha}^A F_\nu^{A\alpha} - \frac{1}{10}g_{\mu\nu}F_{\alpha\beta}^A F^{A\alpha\beta}\Big) = 0.$$

The minimal theory arises as a truncation of type I supergravity compactified on a T^3, or $d = 11$ supergravity compactified on K3 [27].

2.2 The gauged theory

To gauge the theory, one covariantises with respect to $SU(2)$,

$$e^\phi F^A = dA^A + \frac{g}{2}\epsilon^{ABC}A^B\wedge A^C, \tag{2.10}$$

and adds the following terms to the bosonic Lagrangian density:

$$\delta\mathcal{L} = e(-V(\phi) + 8he^{-2\phi}\star(G\wedge A_{(3)})). \tag{2.11}$$

The potential is given by

$$V(\phi) = -60m^2 + 10(m')^2, \tag{2.12}$$

where m is a function of the single scalar field ϕ,

$$m = -\frac{2}{5}he^{-4\phi} - \frac{1}{10}ge^\phi, \tag{2.13}$$

with g the gauge coupling (we have rescaled the coupling in [25] by $g \to g/\sqrt{2}$) and h the (constant) topological mass. One modifies the supersymmetry transformations of the fermions by adding the terms

$$\delta \psi_\mu^a|_{gauge} = m\Gamma_\mu \epsilon^a - igA_{\mu\,b}^a \epsilon^b, \tag{2.14}$$

$$\delta \lambda^a|_{gauge} = -\sqrt{5} m' \epsilon^a. \tag{2.15}$$

In addition to $SU(2)$ covariantisation, the field equations are modified by the addition of the following terms:

$$P|_{gauge} = -V', \tag{2.16}$$

$$Q|_{gauge} = 8he^{-4\phi} \star G, \tag{2.17}$$

$$E_{\mu\nu}|_{gauge} = -\frac{2}{5}g_{\mu\nu}V \tag{2.18}$$

When $h = 0$, the $SU(2)$ theory lifts on an S^3 to the NS sector in $d = 10$ [28]. When $hg > 0$, it lifts on an S^4 to $d = 11$ [29]. When $h \neq 0$, there is a subtlety in imposing the four-form field equation. The reason is that the 3-form $A_{(3)}$, which is massive, would have twenty on-shell degrees of freedom if it satisfied an ordinary second order field equation. However the 3-form in the 7d supergravity multiplet should have only ten on-shell degrees of freedom. This is achieved by imposing the odd-dimensional selfduality equation [30]:

$$e^{2\phi} \star G - \frac{1}{2}(e^\phi F^A \wedge A^A - \frac{g}{6}\epsilon^{ABC}A^A \wedge A^B \wedge A^C) + 8hA_{(3)} = 0. \tag{2.19}$$

Note that the exterior derivative of this equation is just $\star Q$. Imposing the Bianchi identity and $Q = 0$ fixes $A_{(3)}$ up to an arbitrary closed three form. The closed three form is then determined by demanding that $A_{(3)}$ satisfies (2.19). In the examples given below, we will explicitly impose the Bianchi identity and $Q = 0$, but leave the determination of the closed three form in $A_{(3)}$ implicit.

3. Necessary and sufficient conditions for supersymmetry

We present here a set of necessary and sufficient conditions corresponding to having at least one Killing spinor. The reader interested in the derivation of these conditions may consult appendices B-D. We start with the minimal theory and then at the end of the section present the results for the gauged case.

The general metric is given by

$$ds^2 = -H^2(dt + \omega)^2 + g_{ij}dx^i dx^j, \tag{3.1}$$

where there is no t dependence in H, ω, g_{ij}. $\frac{\partial}{\partial t}$ is Killing with associated one-form

$$V = -H^2(dt + \omega). \tag{3.2}$$

There is a natural decomposition over a 6–dimensional Riemannian manifold with metric $g_{ij}dx^i dx^j$ which we will refer to as the base henceforth. The base admits an $SU(3)$ structure, according to appendix B, with an almost complex structure J and a holomorphic 3–form Ω. We may choose the basis on the base such that these take the canonical form

$$J = e^{12} + e^{34} + e^{56}, \tag{3.3}$$

$$\Omega = (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6). \tag{3.4}$$

Redefine the two-form field strengths according to

$$\tilde{F}^a{}_b = (S^{-1}FS)^a{}_b, \tag{3.5}$$

$$K = \tilde{F}^3, \tag{3.6}$$

$$L = \tilde{F}^1 - i\tilde{F}^2, \tag{3.7}$$

where S is the $SU(2)$ matrix given by equations (B.31), (B.32) and the subsequent discussion. Also define

$$C_\mu{}^a{}_b = i(S^{-1})^a{}_c \, \partial_\mu S^c{}_b. \tag{3.8}$$

Then the most general expression for the matter fields of the minimal theory given the existence of a timelike Killing spinor is as follows:

$$\phi = \phi(x) \tag{3.9}$$

$$K = -\frac{1}{H}V \wedge d(\log(He^\phi)) + \frac{1}{4}\Big(i(C^1 - iC^2)\lrcorner\Omega + c.c.\Big) - Hd\omega + \tilde{K}^{(1,1)_0}, \tag{3.10}$$

$$L = -\frac{i}{H}V \wedge (C^1 - iC^2) - \frac{1}{4}(d\log H + 2\mathcal{W}_4)\lrcorner\overline{\Omega} - 2i\overline{\mathcal{W}}_1 J - i\overline{\mathcal{W}}_2 + L^{(2,0)} \tag{3.11}$$

$$G = -\frac{e^{2\phi}}{2H^2}V \wedge d(He^{-2\phi}J)$$
$$+ \star_6\Big(-\frac{H}{2}d\omega + \tilde{K}^{(1,1)_0} + \frac{1}{4}\Big(i(C^1 - iC^2)\lrcorner\Omega + c.c.\Big)\Big). \tag{3.12}$$

The forms are almost entirely fixed in terms of the metric, the dilaton and the $d = 6$ structure. The only unconstrained components are $\tilde{K}^{(1,1)_0}$ and $L^{(2,0)}$, which are arbitrary. The torsion modules, whose definition is given in appendix C, are given by

$$\mathcal{W}_1 = -\frac{i}{6}\overline{L}\lrcorner J, \tag{3.13}$$

$$\mathcal{W}_2 = -i\overline{L}^{(1,1)_0}, \tag{3.14}$$

$$\mathcal{W}_{3ijk} = 2G_{0ijk}^{(2,1)_0}, \tag{3.15}$$

$$\mathcal{W}_{4i} = \frac{1}{2}G_{0ilm}J^{lm} - \frac{e^{2\Phi}}{H}\partial_i\left(He^{-2\Phi}\right), \tag{3.16}$$

$$\mathcal{W}_{5i} = -H^{-1}\partial_i H + \partial_i\Phi + J_i{}^j C_j^3. \tag{3.17}$$

These conditions give the general supersymmetric ansatz for the field equations. Of course since we have a Killing spinor some of the field equations will be identically satisfied. As we show in the appendix E, it is sufficient to impose the Bianchi identities for the forms, and the four form and dilaton field equations. The integrability conditions then guarantee that the remaining field equations are satisfied.

Now we discuss the gauged theory. The metric is the same, and in appendix C we show that there exists a gauge such that the matter fields are given by

$$\phi = \phi(x) \tag{3.18}$$

$$K = -\frac{1}{H} V \wedge d(\log(He^{\phi})) + \frac{1}{4}\Big(ig(A^1 - iA^2)\lrcorner\Omega + c.c.\Big) - Hd\omega + \tilde{K}^{(1,1)_0}$$

$$+ \frac{1}{3}(8he^{-4\phi} + ge^{\phi})J, \tag{3.19}$$

$$L = -\frac{ig}{H} V \wedge (A^1 - iA^2) - \frac{1}{4}(d\log H + 2\mathcal{W}_4)\lrcorner\overline{\Omega} - 2i\overline{\mathcal{W}}_1 J - i\overline{\mathcal{W}}_2 + L^{(2,0)}, \tag{3.20}$$

$$G = -\frac{e^{2\phi}}{2H^2} V \wedge d(He^{-2\phi}J)$$

$$+ \star_6 \Big(-\frac{H}{2}d\omega + \tilde{K}^{(1,1)_0} + \frac{1}{4}\Big(ig(A^1 - iA^2)\lrcorner\Omega + c.c.\Big) - \frac{1}{3}(4he^{-4\phi} - ge^{\phi})J\Big). \tag{3.21}$$

Again there are two arbitrary forms, $\tilde{K}^{(1,1)_0}$ and $L^{(2,0)}$. The first four torsion modules of the six dimensional $SU(3)$ structure are the same as for the ungauged case while the fifth is given by

$$\mathcal{W}_{5i} = -H^{-1}\partial_i H + \partial_i\phi + gJ_i{}^j A_j^3. \tag{3.22}$$

As before, in order to ensure a solution of the field equations, it is sufficient to impose the Bianchi identities, the four form field equation and the dilaton field equation. Now we turn to an investigation of the supersymmetric solutions of the theories.

4. Supersymmetric solutions of the minimal theory

In this section we will employ our general supersymmetric ansatz for the minimal theory to determine some new solutions. The stategy is to make a choice for the base which satisfies the required constraints on the torsion, and to use the field equations to determine the field components which are not fixed by the (six dimensional) torsion. The general problem is still complicated, so we will restrict attention to some specific types of base.

4.1 Calabi-Yau base

The simplest choice for the base is to take it to be Calabi-Yau. Rather trivially, we see that all the vacuum solutions of the theory are of the form $\mathbb{R} \times CY_6$. More interestingly, we may obtain Gödel solutions by looking for solutions of the form

$L = \tilde{K}^{(1,1)_0} = \phi = 0$, $H = 1$[1]. The Bianchi identity for G then implies that $d\omega$ is coclosed, and it is a simple matter to check that the dilaton and four form equations of motion are satisfied. Explicitly, one might choose a flat base with metric $\delta_{ij} dx^i dx^j$, and ω of the form

$$\omega = \alpha \sum_{n=0}^{2} (x^{2n+1} dx^{2n+2} - x^{2n+2} dx^{2n+1}), \tag{4.1}$$

and the fluxes are

$$K = -2\alpha J, \tag{4.2}$$

$$G = -\frac{1}{2} J \wedge J. \tag{4.3}$$

There are closed timelike curves for $(x^{2n+1})^2 + (x^{2n+2})^2 > \alpha^{-2}$. More generally, we could have included $(2,0) + (0,2)$ and $(1,1)_0$ parts in $d\omega$.

4.2 Semi-Kähler base

A semi-Kähler base is one for which the only nonvanishing torsion module is $\mathcal{W}_3$. We will seek solutions induced by such a base with $H = 1$, $\phi = 0$. The forms then reduce to

$$G = -\frac{1}{2} V \wedge \mathcal{W}_3 + \star_6 (-\frac{1}{2} d\omega + \tilde{K}^{(1,1)_0}), \tag{4.4}$$

$$K = -d\omega + \tilde{K}^{(1,1)_0}, \tag{4.5}$$

$$L = L^{(2,0)}. \tag{4.6}$$

An explicit example of a semi-Kähler threefold is the three dimensional complex Heisenberg group. It admits a left-invariant metric with structure equations

$$de^a = 0, \quad a = 1, .., 4, \tag{4.7}$$

$$de^5 = e^{13} - e^{24}, \tag{4.8}$$

$$de^6 = e^{14} + e^{23}, \tag{4.9}$$

so that

$$dJ = e^{136} - e^{246} - e^{145} - e^{235} = \mathcal{W}_3, \tag{4.10}$$

$$d\Omega = 0. \tag{4.11}$$

Now, defining

$$\{x^i, x^j\} = \frac{1}{2}(x^i dx^j - x^j dx^i), \tag{4.12}$$

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[1] When H and ϕ are constant, we may without any essential loss of generality set $H = 1$, $\phi = 0$ in both the minimal and gauged theories by performing suitable rescalings.

we may introduce coordinates according to

$$e^a = dx^a, \quad a = 1, .., 4, \tag{4.13}$$

$$e^5 = du + \{x^1, x^3\} - \{x^2, x^4\}, \tag{4.14}$$

$$e^6 = dv + \{x^1, x^4\} + \{x^2, x^3\}. \tag{4.15}$$

The Bianchi identities for the two forms imply that $\tilde{K}^{(1,1)_0}$ and $L^{(2,0)}$ are closed. We will choose

$$\tilde{K}^{(1,1)_0} = \alpha(e^{13} + e^{24}) + \beta(e^{14} - e^{23}), \tag{4.16}$$

$$L^{(2,0)} = \gamma[e^{13} - e^{24} + i(e^{14} + e^{23})], \tag{4.17}$$

for constant α, β, γ, and γ complex. The four form Bianchi identity may then be solved by taking

$$\tilde{K} = d\omega^{(1,1)_0}, \tag{4.18}$$

$$d\omega^{(0,0)} = 0, \tag{4.19}$$

$$d(d\omega^{(2,0)+(0,2)}) = 0, \tag{4.20}$$

and so we may choose

$$d\omega^{(2,0)+(0,2)} = \delta(e^{13} - e^{24}) + \epsilon(e^{14} + e^{23}), \tag{4.21}$$

for constant δ, ϵ. Hence ω is given by

$$\omega = (\alpha + \delta)\{x^1, x^3\} + (\alpha - \delta)\{x^2, x^4\} + (\beta + \epsilon)\{x^1, x^4\} + (-\beta + \epsilon)\{x^2, x^3\}. \tag{4.22}$$

The four form field equation imposes

$$\alpha^2 + \beta^2 = 2(|\gamma|^2 + 1), \tag{4.23}$$

and it may be verified that this is equivalent to the dilaton field equation, so there are no further constraints.

4.3 Hermitian base

Manifolds in the class $\mathcal{W}_3 \oplus \mathcal{W}_4$ are called Hermitian, as they have vanishing Nijenhuis tensor. Again we consider solutions for which $H = 1$, $\phi = 0$, $C = 0$. Then we have

$$K = -d\omega + \tilde{K}^{(1,1)_0}, \tag{4.24}$$

$$L = -\frac{1}{2}\mathcal{W}_4 \lrcorner \overline{\Omega} + L^{(2,0)}, \tag{4.25}$$

$$G = \frac{1}{2}(dt + \omega) \wedge dJ + \star_6 \left(-\frac{1}{2}d\omega + \tilde{K}^{(1,1)_0} \right). \tag{4.26}$$

As an example we take the fibration of a two–dimensional flat space (a torus or a plane) over another four–dimensional flat space:

$$de^5 = e^{12}, \tag{4.27}$$

$$de^6 = e^{34}, \tag{4.28}$$

or equivalently, defining the one–forms $z^1 = e^1 + ie^2$, $z^2 = e^3 + ie^4$, $z^3 = e^5 + ie^6$,

$$dz^3 = \frac{1}{2}\left(iz^{1\bar{1}} - z^{2\bar{2}}\right). \tag{4.29}$$

A direct check shows that

$$dJ = e^{126} - e^{345}, \tag{4.30}$$

$$\mathcal{W}_3 = \frac{1+i}{8}\left(z^{1\bar{1}} - z^{2\bar{2}}\right)z^3, \tag{4.31}$$

$$\mathcal{W}_4 = \frac{1}{2}\left(e^6 - e^5\right). \tag{4.32}$$

The Bianchi identities for the two forms imply they are closed. We choose

$$\tilde{K}^{(1,1)_0} = \beta_1 z^{1\bar{2}} + \bar{\beta}_1 z^{\bar{1}2} - iB_2\left(z^{1\bar{1}} - z^{2\bar{2}}\right), \tag{4.33}$$

$$L = \frac{1+i}{4}z^{\bar{1}2} + \alpha z^{12}, \tag{4.34}$$

where the constants β_1, α are complex and B_2 real. We also choose the form of ω to be

$$\omega = C_1 e^1 + C_2 e^2 + C_3 e^3 + C_4 e^4 + C_5 e^5 + C_6 e^6, \tag{4.35}$$

with the C_i real. Then the Bianchi identity for G gives $\beta_1 = 0$, $B_2 = 1/4\,(C_6 - C_5)$. Lastly, the dilaton and four–form field equations are equivalent to

$$\frac{1}{8} + |\alpha|^2 = \frac{1}{8}(C_5 - C_6)^2, \tag{4.36}$$

$$|\alpha|^2 = \frac{3}{8} - \frac{1}{8}(C_5 - C_6)^2, \tag{4.37}$$

with solution $|C_6 - C_5| = \sqrt{2}$, $|\alpha| = 1/2\sqrt{2}$. Taking for example $C_6 > C_5$ and $\alpha = \frac{1+i}{4}e^{i\theta}$ we get the final form

$$K = -\left(C_5 + \frac{1}{\sqrt{2}}\right)\left(e^{12} + e^{34}\right), \tag{4.38}$$

$$L = \frac{1+i}{4}\left[(1 + e^{i\theta})(e^{13} - e^{24}) - i(1 - e^{i\theta})(e^{23} + e^{14})\right], \tag{4.39}$$

$$G = \frac{1}{2}(dt + \omega) \wedge (e^{126} - e^{345}) - \frac{1}{2}\left[C_5 e^{12} + (C_5 + \sqrt{2})e^{34}\right] \wedge e^{56}, \tag{4.40}$$

$$\omega = C_1 e^1 + C_2 e^2 + C_3 e^3 + C_4 e^4 + C_5 e^5 + (C_5 + \sqrt{2})e^6. \tag{4.41}$$

This provides a six-parameter family of solutions.

5. Supersymmetric solutions of the gauged theory

Our choice of gauge in the gauged theory was motivated by the desire that the metric, structure and matter fields should all be preserved along V. However we have seen this choice breaks manifest $SU(2)$ covariance, and also that any timelike supersymmetric solutions with non-zero Yang-Mills fields necessarily involve gauge fields with electric components. These points have hampered our efforts to obtain explicit Yang-Mills solutions, so instead we will focus on the abelian case. To obtain $U(1)$ solutions, we set $L = 0$. This implies the following constraints on the torsion of the base:

$$\mathcal{W}_1 = \mathcal{W}_2 = 0, \tag{5.1}$$

$$\mathcal{W}_4 = -\frac{1}{2} d\log H. \tag{5.2}$$

Equation (5.1) implies that the complex structure on the base is integrable. Next, conformally rescaling the base according to $g_6 = H^{-1/2}\tilde{g}_6$, $J = H^{-1/2}\tilde{J}$, $\Omega = H^{-3/4}\tilde{\Omega}$, we see that $\tilde{J}, \tilde{\Omega}$ define a canonical complex structure on the base with metric $\tilde{g}$, and

$$\tilde{\mathcal{W}}_1 = \tilde{\mathcal{W}}_2 = \tilde{\mathcal{W}}_4 = 0, \tag{5.3}$$

$$\tilde{\mathcal{W}}_5 = -\frac{1}{4} d\log H + d\phi - i_{A^3} J. \tag{5.4}$$

For the remainder of this section we will drop the tildes. So far, this is the general $U(1)$ ansatz. However we will now restrict to solutions of the specific form

$$\mathcal{W}_3 = 0, \tag{5.5}$$

together with $\phi = 0$, $H = 1$. Then the base is Kähler, and $i_V G = 0$. Furthermore, (5.4) then implies that $-gA_i^3$ is the potential for the Ricci form on the base, ie

$$\mathfrak{R}_{ij} = \frac{1}{2} R_{ijkl} J^{kl} = -gdA_{ij}^3 \tag{5.6}$$

Then the Bianchi identity for K is equivalent to

$$\mathfrak{R} = -\frac{1}{3}(8gh + g^2)J - g\tilde{K}^{(1,1)_0}. \tag{5.7}$$

Since the scalar curvature of the base is $R = 2J\lrcorner\mathfrak{R}$, we see that for $hg \geq 0$ we must choose a constant negative scalar curvature base, and the choice of base determines $\tilde{K}$. The forms may now be written as

$$K = -d\omega - \frac{\mathfrak{R}}{g}, \tag{5.8}$$

$$G = -\star_6 \left(\frac{d\omega}{2} + \frac{\mathfrak{R}}{g} + 4hJ\right) \tag{5.9}$$

Since both $\mathfrak{R}$ and J are closed and coclosed in this context, the Bianchi identity and field equation for G are

$$d \star d\omega = 0, \tag{5.10}$$

$$2h(d\omega^{(0,0)} + 4d\omega^{(1,1)_0}) \wedge J = \frac{1}{2g^2}\mathfrak{R} \wedge \mathfrak{R} - \frac{8h}{g}\mathfrak{R} \wedge J$$
$$- (16h^2 + 4hg)J \wedge J. \tag{5.11}$$

When the topological mass is zero, (5.11) implies that we must choose the base such that its Ricci form is decomposable. When $h \neq 0$, (5.11) determines the (0,0) and $(1,1)_0$ parts of $d\omega$ in terms of the geometry of the base (note that the $(2,0) + (0,2)$ part drops out), and we must then impose (5.10). Finally, the dilaton field equation reads

$$-4G \lrcorner G + K \lrcorner K + (8h - g)(16h - g) = 0. \tag{5.12}$$

Within our restricted ansatz, the only place (2,0)+(0,2) forms can arise is through $d\omega$. As we have seen, these components drop out of four form field equations. In fact they also drop out of (5.12). Furthermore since $\star d\omega^{(2,0)+(0,2)} = d\omega^{(2,0)+(0,2)} \wedge J$, any solution of our system of equations may be deformed by the addition of an arbitrary closed (2,0)+(0,2) form to $d\omega$.

5.1 Examples: h=0

When the topological mass vanishes, we must choose the base such that the Ricci form is decomposable and $R = -2g^2$. An example is $\mathcal{M}_4 \times H^2$, where $\mathcal{M}_4$ is any hyperkähler manifold. The base has metric

$$ds^2 = (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 + \frac{1}{g^2}(d\theta^2 + \sinh^2 \theta d\psi^2), \tag{5.13}$$

and the Ricci form is $\mathfrak{R} = -\sinh \theta d\theta \wedge d\psi \equiv -g^2 e^5 \wedge e^6$. We may solve (5.10) by taking

$$\omega = \alpha \cosh \theta d\psi. \tag{5.14}$$

Then (5.12) implies that

$$\alpha = g^{-1}. \tag{5.15}$$

By rescaling t, we may express the full solution as

$$ds^2 = \frac{1}{g^2}\left(-(\Sigma^3)^2 + (\Sigma^1)^2 + (\Sigma^2)^2\right) + ds^2(\mathcal{M}_4), \tag{5.16}$$

$$G = \frac{g}{2}e^{1234}, \quad K = 0, \tag{5.17}$$

where Σ^i are the invariant one forms on AdS_3 (see appendix A for details).

A second example of a solution with this base may be obtained by taking $d\omega = gJ$, and the metric is

$$ds^2 = -(dt + g\rho + g^{-1} \cosh\theta d\psi)^2 + ds^2(\mathcal{M}_4) + g^{-2}(d\theta^2 + \sinh^2\theta d\psi^2), \qquad (5.18)$$

where ρ is the Kähler form potential on $\mathcal{M}_4$, $d\rho = J_4$. These solutions have closed timelike curves irrespective of the choice of $\mathcal{M}_4$, as may be seen from the norm of $\frac{\partial}{\partial\psi}$.

5.2 Examples: $hg > 0$

When the topological mass is nonzero, we have a wider range of allowed bases than when $h = 0$; in particular, the restrictive condition $\mathfrak{R} \wedge \mathfrak{R} = 0$ is lifted. We will now consider some examples.

Vacuum solutions

Imposing $K = G = 0$ on our ansatz, we may deduce that $d\omega = -g^{-1}\mathfrak{R} = 8hJ$, and hence $16h = g$. We must thus choose an Einstein base with scalar curvature $R = -3g^2$, and taking $d\omega = \frac{g}{2}J$ implies that the seven dimensional solution is also Einstein. For instance, it is well known [32] that all $2n + 1$-dimensional AdS spaces may be obtained as circle bundles over a complex hyperbolic n-space $\mathcal{H}_\mathbb{C}^n$ equipped with its Bergman metric. Explicitly, we choose the base with metric

$$ds^2 = \frac{4}{g^2}\Big[dr^2 + \frac{1}{4}\sinh^2 r(\Sigma^3 - \sigma^3)^2 + \cosh^2(\frac{r}{2})\Big((\Sigma^1)^2 + (\Sigma^2)^2\Big) + \sinh^2(\frac{r}{2})\Big((\sigma^1)^2 + (\sigma^2)^2\Big)\Big],$$
$$(5.19)$$

and we then find that

$$(dt + \omega) = \frac{1}{g}(\Sigma^3 + \sigma^3 + \cosh r(\Sigma^3 - \sigma^3)), \qquad (5.20)$$

thus obtaining a metric on AdS_7. In a similar fashion we may obtain other seven dimensional Einstein manifolds admitting Killing spinors. Some examples are circle bundles over $H^2 \times H^2 \times H^2$ or $\mathcal{H}_\mathbb{C}^2 \times H^2$, with metrics

$$ds^2 = \frac{1}{g^2}\Big[-(dt + \sum_1^3 \cosh r_i d\phi_i)^2 + 2\sum_1^3(dr_i^2 + \sinh^2 r_i d\phi_i^2)\Big], \qquad (5.21)$$

$$ds^2 = \frac{1}{g^2}\Big[-\frac{1}{4}(dt - 3\sinh^2 r\sigma^3 + 2\cosh\theta d\phi)^2 + 2\Big(d\theta^2 + \sinh^2\theta d\phi\Big)$$
$$+ \frac{4}{3}\Big(dr^2 + \frac{1}{4}\sinh^2 r((\sigma^1)^2 + (\sigma^2)^2 + \cosh^2 r(\sigma^3)^2)\Big)\Big]. \qquad (5.22)$$

These both have closed timelike curves, which in contrast to AdS may not be eliminated by going to the covering space. Such solutions have been discussed previously in eg. [33].

Squashed AdS_7

Since $\mathcal{H}_{\mathbb{C}}^3$ equipped with its Bergman metric is Einstein, we have $\tilde{K}^{(1,1)_0} = 0$, and so from (5.11) $d\omega^{(1,1)_0} = 0$ when we choose this base. Allowing for non-zero fluxes on the base, the (0,0) part of $d\omega$ will change to give a squashed fibration.

To see this explicitly, consider $d\omega^{(2,0)+(0,2)} = 0$, $d\omega \sim J$, and for simplicity, $K = 0$. We obtain a solution provided that $4h = g$, $\mathfrak{R} = -g^2 J = -g d\omega$, and $G = -1/4 J \wedge J$. In terms of the AdS siebenbeins $e^0 = g^{-1}(\Sigma^3 + \sigma^3 + \cosh r (\Sigma^3 - \sigma^3))$, $e^1 = 2g^{-1}dr$, etc, the metric is

$$ds^2 = -(e^0)^2 + \frac{1}{2}\delta_{ij}e^i e^j \tag{5.23}$$

We may similarly squash the other vacuum solutions given above.

Product base: AdS_5 **and** AdS_3 **solutions**

When we choose the base to be a product of two or three Kähler manifolds, we generically find upon solving for ω that the seven dimensional solution has closed timelike curves. However if we take the base to be of the form $\mathcal{H}_{\mathbb{C}}^2 \times \mathcal{M}_2$ or $H^2 \times \mathcal{M}_4$, it is possible to arrange the curvatures in such a way that we can obtain $d\omega \sim J_4$ or $d\omega \sim J_2$, and thus obtain $AdS_5 \times \mathcal{M}_2$ or $AdS_3 \times \mathcal{M}_4$ solutions.

The simplest way to determine the AdS_5 solution is to impose $G = 0$, $K \sim vol(\mathcal{M}_2)$, and seek a base of the form

$$ds^2 = a^2 \left(dr^2 + \frac{1}{4}\sinh^2 r ((\sigma^1)^2 + (\sigma^2)^2 + \cosh^2 r (\sigma^3)^2) \right) + ds^2(\mathcal{M}_2), \tag{5.24}$$

together with $d\omega = 2a^{-1}J_4$. One may indeed find such a solution, provided that $g = 12h$, $a = 3g^{-1}$, $\mathcal{M}_2 = H^2$ with scalar curvature $-2g^2/3$, and $K = g/3 J_2$. This is one of the AdS_5 solutions given by Maldacena and Nunez in [1]. As in [14], one may deform the AdS_5, by the addition of $(2,0) + (0,2)$ terms to $d\omega$. Explicitly, one may add the terms

$$\alpha d(\tanh^2 r \sigma^1) + \beta d(\tanh^2 r \sigma^2), \tag{5.25}$$

for constant α, β.

To obtain the AdS_3 solutions, we take the base to be of the form

$$ds^2 = a^2(dr^2 + \sinh^2 r d\phi^2) + ds^2(\mathcal{M}_4). \tag{5.26}$$

We seek solutions with $d\omega = a^{-1}J_2$, and require that neither K nor G have components on the H^2. We find an $AdS_3 \times \mathcal{M}_4$ solution provided that $a = g^{-1}$, $g = 12h$, $\mathcal{M}_4$ is negative scalar curvature Kähler-Einstein with $\mathfrak{R}_4 = -g^2/3 J_4$, and the forms are

$$K = \frac{g}{3}J_4, \tag{5.27}$$

$$G = \frac{g}{12}J_4 \wedge J_4. \tag{5.28}$$

This solution describes the AdS fixed point of the near-horizon limit of an M5 brane wrapped on a Kähler four cycle in a Calabi-Yau four-fold [34].

6. Conclusions

In this work we have studied bosonic field configurations of minimal and $SU(2)$ gauged $d = 7$ supergravity admitting timelike Killing spinors, and shown that such a spinor is equivalent to an $SU(3)$ structure. We have given necessary and sufficient conditions for its existence and hence obtained the most general superymmetric ansatz for the theories. The bosonic fields are largely determined by the structure, but the structure itself, before imposing the field equations, is weakly constrained. We have exploited the general ansatz to explicitly present numerous new solutions.

One of the more striking features we have found, in common with other G-structure oriented studies of timelike Killing spinors, is the apparent genericity of spacetimes with closed timelike curves among supersymmetric solutions. In the exceptional case of the AdS spaces, the CTCs may be eliminated by going to the universal cover. In other cases, such as the Gödel solutions of the minimal theory, the CTCs may be eliminated by taking a suitable quotient. Explicitly, taking the flat base of section 4.1 to be toroidally compactified such that

$$(x^{2n+1})^2 + (x^{2n+2})^2 < \alpha^{-2}, \tag{6.1}$$

eliminates the CTCs. More generally, it would be interesting to know to what extent one might be able to eliminate CTCs in all metrics of the form (3.1) by taking an appropriate quotient or covering. If a class of such spacetimes exist for which this is not possible, one would be forced to invoke some dynamical chronology protection agent in string theory.

In this work we have only begun to explore the solutions contained in the general ansatz, and have restricted attention to particularly simple constructions. In the gauged theory we restricted attention to a $U(1)$ truncation with Kähler base and constant H, ϕ. In particular, we have not explicitly given any Yang-Mills solutions. Clearly, there is scope for a more systematic study of restrictions of the general ansatz, which we have not pursued here.

It is known that the gauged theory admits Yang-Mills solutions describing the near-horizon limits of branes wrapped on various supersymmetric cycles. We expect these solutions to be contained in the null class, where the G-structure defined by a single Killing spinor is $(SU(2) \ltimes \mathbb{R}^4) \times \mathbb{R}$, which naturally induces a $2 + 1 + 4$ split of the seven dimensional spacetime. It will be interesting to analyse this case in more detail, and in particular, we hope to systematically undertake a more refined classification.

Acknowledgements

A. Conventions

We work in mostly plus signature. Indices in 7 dimensions are given by $\mu, \nu, \ldots$, in 6 dimensions by $i, j, \ldots$. The Dirac algebra is

$$\{\Gamma_\mu, \Gamma_\nu\} = 2g_{\mu\nu} \,. \tag{A.1}$$

This tells us that in an orthonormal frame Γ_0 is antihermitian and the Γ_i $(i = 1, \ldots, 6)$ are hermitian. Following the appendix to Chapter 1 in [24] we have that the charge conjugation matrix C satisfies

$$C^T = C \,, \qquad C^\dagger C = \boldsymbol{I} \,, \qquad \Gamma_\mu^T = -C\Gamma_\mu C^{-1} \,. \tag{A.2}$$

We can therefore choose

$$C = \boldsymbol{I} \,. \tag{A.3}$$

This implies that Γ_0 is real and the Γ_i are imaginary. We will choose a representation (there are two inequivalent ones) such that

$$\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\Gamma_6 = -\boldsymbol{I} \,. \tag{A.4}$$

We also have the identity

$$\Gamma_{\alpha_1 \ldots \alpha_n} = \frac{(-)^{[n/2]+1}}{(7-n)!} \epsilon_{\alpha_1 \ldots \alpha_n \beta_1 \ldots \beta_{7-n}} \Gamma^{\beta_1 \ldots \beta_{7-n}} \,. \tag{A.5}$$

We choose the orientation to be given by $\epsilon^{0123456} = +1$.

The Dirac conjugate $\bar{\epsilon}_a$ of an anticommuting spinor ϵ^a is defined as

$$\bar{\epsilon}_a = (\epsilon^a)^\dagger \Gamma_0 \,, \tag{A.6}$$

and we also define

$$\bar{\epsilon}^a = \epsilon^{ab} \bar{\epsilon}_b \,, \tag{A.7}$$

where ϵ^{ab} is a constant antisymmetric matrix satisfying $\epsilon_{ab} \epsilon^{bc} = -\delta_a^c$ that is used to raise and lower spinor indices according to $\epsilon^a \equiv \epsilon^{ab}\epsilon_b$, and $\epsilon^{12} = 1$. On the other hand the symplectic-Majorana conjugate ϵ^C of ϵ is defined to be

$$(\epsilon^C)^a = (\epsilon^T)_b \,. \tag{A.8}$$

Symplectic-Majorana spinors are those for which (A.7) is equal to (A.8), namely

$$(\epsilon^T)^a = \bar{\epsilon}^a \,. \tag{A.9}$$

Given four spinors $\epsilon_1, \ldots, \epsilon_4$, the Fierz identity is

$$\overline{\epsilon_1}\epsilon_2\overline{\epsilon_3}\epsilon_4 = \frac{1}{8}\left[\overline{\epsilon_1}\epsilon_4\overline{\epsilon_3}\epsilon_2 + \overline{\epsilon_1}\Gamma_\mu\epsilon_4\overline{\epsilon_3}\Gamma^\mu\epsilon_2 - \frac{1}{2}\overline{\epsilon_1}\Gamma_{\mu\nu}\epsilon_4\overline{\epsilon_3}\Gamma^{\mu\nu}\epsilon_2 - \frac{1}{3!}\overline{\epsilon_1}\Gamma_{\mu\nu\rho}\epsilon_4\overline{\epsilon_3}\Gamma^{\mu\nu\rho}\epsilon_2\right].$$

(A.10)

At various points in the text we make use of invariant one forms on S^3 and AdS_3. In terms of the Euler angles (θ, ϕ, ψ) the right invariant one forms are given by

$$\sigma^1 = \sin\psi d\theta - \cos\psi\sin\theta d\phi,$$ (A.11)

$$\sigma^2 = \cos\psi d\theta + \sin\psi\sin\theta d\phi,$$ (A.12)

$$\sigma^3 = d\psi + \cos\theta d\phi.$$ (A.13)

These obey $d\sigma^A = -\frac{1}{2}\epsilon^{ABC}\sigma^B \wedge \sigma^C$, and one may write the round $SU(2)$ invariant metric on a unit S^3 as

$$ds^2 = \frac{1}{4}\delta_{AB}\sigma^A\sigma^B.$$ (A.14)

By analytically continuing $\theta \to i\theta$, extending the range of θ to $[0, \infty)$ and changing the sign of the metric, one may obtain an $SL(2, \mathbb{R})$ invariant metric on a unit AdS_3 as

$$ds^2 = \frac{1}{4}\eta_{AB}\Sigma^A\Sigma^B,$$ (A.15)

where

$$\Sigma^1 = \sin\psi d\theta - \cos\psi\sinh\theta d\phi,$$ (A.16)

$$\Sigma^2 = \cos\psi + \sin\psi\sinh\theta d\phi,$$ (A.17)

$$\Sigma^3 = d\psi + \cosh\theta d\phi.$$ (A.18)

The Σ^A obey

$$d\Sigma^1 = -\Sigma^2 \wedge \Sigma^3, \quad d\Sigma^2 = -\Sigma^3 \wedge \Sigma^1,$$
$$d\Sigma^3 = \Sigma^1 \wedge \Sigma^2.$$

(A.19)

The metric (A.15) clearly has closed timelike curves for all constant (θ, ψ). These may be eliminated by going to the universal cover: define

$$\psi = u + v,$$ (A.20)

$$\phi = u - v,$$ (A.21)

$$\theta = 2r.$$ (A.22)

Then (A.15) becomes

$$ds^2 = -\cosh^2 r du^2 + dr^2 + \sinh^2 r dv^2,$$ (A.23)

and taking the ranges $-\infty < u < \infty$, $0 \leq v < 2\pi$, we have the familiar global metric on the universal cover of AdS_3.

Finally we note the following useful identity for a two form A on a six dimensional manifold equipped with an $SU(3)$ structure:

$$A \lrcorner (J \wedge J) = 4A^{(0,0)} + 2A^{(2,0)+(0,2)} - 2A^{(1,1)_0} \tag{A.24}$$

By taking the dual of this equation we may deduce that

$$\star A^{(0,0)} = \frac{1}{2}A^{(0,0)} \wedge J, \tag{A.25}$$

$$\star A^{(2,0)+(0,2)} = A^{(2,0)+(0,2)} \wedge J, \tag{A.26}$$

$$\star A^{(1,1)_0} = -A^{(1,1)_0} \wedge J. \tag{A.27}$$

B. Bilinears and the G-Structure

The standard strategy in applying G-structures to the solution of supergravities is to assume the existence of a (globally defined) Killing spinor. The existence of a globally defined spinor is equivalent to the existence of a set globally defined forms, constucted as bilinears in the spinor, which are invariant under the isotropy group, G, of the spinor. This in turn implies a global reduction of the principal frame bundle with structure group $Spin(1,6)$, in the present context, to a subbundle with structure group G. There are two maximal subgroups of $Spin(1,6)$ which leave a spinor invariant, depending on whether the associated vector is timelike or null. As we shall see below, in the timelike case of interest to us here, a Killing spinor defines an $SU(3)$ structure. In the null case, which we leave for future study, one has an $(SU(2) \ltimes \mathbb{R}^4) \times \mathbb{R}$ structure. Here we shall see how the $SU(3)$ structure arises.

Thus, assume there exists a globally defined Killing spinor ϵ^a satisfying the spinor equation $\delta\lambda^a = 0$ and the Killing equation $\delta\psi_\mu^a = 0$. We can define the following spinor bilinears

$$f^{(ab)} = \bar{\epsilon}^a \epsilon^b, \tag{B.1}$$

$$\epsilon^{ab} V_\mu = \bar{\epsilon}^a \Gamma_\mu \epsilon^b, \tag{B.2}$$

$$\epsilon^{ab} I_{\mu\nu} = \bar{\epsilon}^a \Gamma_{\mu\nu} \epsilon^b, \tag{B.3}$$

$$\Omega_{\mu\nu\rho}^{(ab)} = \bar{\epsilon}^a \Gamma_{\mu\nu\rho} \epsilon^b. \tag{B.4}$$

From the reality properties of the gamma matrices and the symplectic Majorana condition, the vector V_μ and the two-form $I_{\mu\nu}$ are seen to be real, while instead the scalars and the 3–form can be rewritten as

$$f^a{}_b = -ig^A (T^A)^a{}_b, \tag{B.5}$$

$$\Omega^a{}_b = -iX^A (T^A)^a{}_b, \tag{B.6}$$

with g^A, $X^A_{\mu\nu\rho}$, $A = 1,2,3$, real. $(T^A)^a{}_b = 1/2(\sigma^A)^a{}_b$ are generators of the $SU(2)$ Lie algebra, σ^A being the Pauli matrices, and obey

$$(T^A)^a{}_b (T^B)^b{}_c = \frac{1}{4}\delta^{AB} \delta^a_c + \frac{i}{2}\epsilon^{ABC} (T^C)^a{}_c. \tag{B.7}$$

One important consequence of the Fierz identity (A.10) is that V_μ is either time-like or null

$$V^2 = -\frac{1}{4} g^A g^A. \tag{B.8}$$

Here we focus on the timelike case only.

Let us introduce coordinates adapted to our timelike Killing vector. We take $V = \frac{\partial}{\partial t}$, and write the general metric admitting a timelike Killing vector as

$$ds^2 = -H^2(dt + \omega)^2 + g_{ij}dx^i dx^j \tag{B.9}$$

where H, ω and g_{ij} are independent of t. As a form, $V = -H^2(dt + \omega) = -He_0$. The chirality matrix on the base is given by

$$\Gamma_* = \Gamma_1 \ldots \Gamma_6, \tag{B.10}$$

and it is equal to $H^{-1}V^\mu\Gamma_\mu = \Gamma_0$, according to (A.4). The Fierz identities for a symplectic-Majorana spinor ϵ^a imply the following projection:

$$\Gamma_* \epsilon^a = \frac{1}{H} f^a{}_b \epsilon^b. \tag{B.11}$$

When V is timelike the spacetime decomposes along a 6–dimensional Riemannian base and the symplectic-Majorana Killing spinor defines an $SU(3)$ structure. We can decompose the bosonic quantities according to the structure. In order to do this, first of all notice that in 6 Riemannian dimensions a Weyl spinor η of unit norm, satisfying

$$\overline{\eta}\eta = 1, \tag{B.12}$$

$$\Gamma_*\eta = i\eta, \tag{B.13}$$

defines a canonical $SU(3)$ structure with a 2–form J and a 3–form Ω given by

$$J_{ij} = i\overline{\eta}\,\Gamma_{ij}\,\eta, \tag{B.14}$$

$$\Omega_{ijk} = \overline{\eta}\,\Gamma_{ijk}\,\eta^\star. \tag{B.15}$$

It is useful to note the projections

$$\Gamma^i\eta = \frac{1}{2}\left(\delta^i_j + iJ^i{}_j\right)\Gamma^j\eta, \tag{B.16}$$

$$\Gamma^{ij}\eta = -\frac{1}{2}\overline{\Omega}^{ijk}\Gamma_k\eta^* - iJ^{ij}\eta, \tag{B.17}$$

that imply among other relations

$$J_{ij}\Gamma^{ij}\eta = -6i\eta, \tag{B.18}$$

$$\Omega_{ijk}\Gamma^{ijk}\eta = -48\eta^\star \tag{B.19}$$

$$J_{il}J_{jm}\Gamma^{lm}\eta = -\Gamma_{ij}\eta - 2iJ_{ij}\eta. \tag{B.20}$$

We may choose our basis so that J and Ω take the standard form

$$J = e^{12} + e^{34} + e^{45}, \tag{B.21}$$

$$\Omega = (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6). \tag{B.22}$$

Now consider a timelike symplectic-Majorana spinor ϵ^a in 7–dimensions. It will admit, for each value of $a = 1, 2$, a chiral decomposition over the base. We may write the most general expression for ϵ^1 in terms of two orthogonal unit norm Dirac spinors η, $\hat{\eta}$ of positive chirality in six dimensions as

$$\epsilon^1 = \epsilon_+^1 + \epsilon_-^1 = f\eta + g\left(\alpha\eta + \beta\hat{\eta}\right)^*, \tag{B.23}$$

where f, g, α, β are functions, $|\alpha|^2 + |\beta|^2 = 1$, and f and g can be taken to be real functions because we are free to redefine η, $\hat{\eta}$ by an arbitrary phase (see [16]). Imposing the symplectic Majorana condition (A.9), using $\Gamma_0\eta = i\eta$, $\Gamma_0\hat{\eta} = i\hat{\eta}$ then implies that ϵ^2 is given by

$$\epsilon^2 = ig\left(\alpha\eta + \beta\hat{\eta}\right) - if\eta^*. \tag{B.24}$$

Now one can calculate the scalars $f^a{}_b$ and get in particular

$$f^1{}_1 = i\left(f^2 - g^2\right), \tag{B.25}$$

$$f^1{}_2 = 2fg\,\alpha^*. \tag{B.26}$$

Using the projection (B.11) one finds the following set of equations:

$$(H + f^2 - g^2)g\beta = 0, \tag{B.27}$$

$$(H - g^2 - f^2)g\alpha = 0, \tag{B.28}$$

$$(H + g(1 - 2|\alpha|^2) - f^2)f = 0, \tag{B.29}$$

$$fg\alpha\beta = 0. \tag{B.30}$$

When $\beta = 0$, and thus $|\alpha|^2 = 1$, there is the condition $f^2 + g^2 = H$. Therefore we may set $f = H^{1/2}\cos\theta$, $g = H^{1/2}\sin\theta$. The angle θ ranges in $[0, \pi]$, since for $\beta = 0$ in (B.24) a change $\theta \to \theta + \pi$ can be reabsorbed by redefining $\eta \to -\eta$. Set also $\alpha = e^{-i\gamma}$, then (B.24) can be rewritten as

$$\epsilon^1 = H^{1/2}\left(\cos\theta\eta + \sin\theta e^{i\gamma}\eta^*\right), \tag{B.31}$$

$$\epsilon^2 = H^{1/2}\left(i\sin\theta e^{-i\gamma}\eta - i\cos\theta\eta^*\right). \tag{B.32}$$

Thus $\epsilon^a = H^{1/2}S^a{}_b\eta^b$, where $S \in SU(2)$ and $\eta^1 = \eta$, $\eta^2 = -i\eta^*$. The $g = 0$ solution of equations (B.27)-(B.29) is clearly the special case $\theta = 0$. When $\alpha = 0$, either g or f are zero for non-zero H, and this is again a special case of $\beta = 0$. Finally, there is a solution of (B.27)-(B.29) with $f = 0$, $g = H^{1/2}$, $|\alpha|^2 + |\beta|^2 = 1$. Together, η and

$\hat{\eta}$ define an $SU(2)$ structure. However, $\tilde{\eta}^* = (\alpha\eta + \beta\hat{\eta})^*$ defines an $SU(3)$ structure for which we may write the corresponding $\epsilon^{1,2}$ in the form (B.32). Therefore the most general timelike Killing spinor may be written in this form, and thus defines an $SU(3)$ structure. Such a spinor will generically preserve a single supersymmetry. Now noting that $i_V I = 0$, $i_V X^A = g^A I$, employing the projections satisfied by η, we may deduce the following form for the bilinears:

$$g^1 - ig^2 = 2Hi\sin 2\theta e^{i\gamma}, \;\; g^3 = -2H\cos 2\theta, \tag{B.33}$$

$$I = HJ, \tag{B.34}$$

$$X^1 - iX^2 = -H^{-1}(g^1 - ig^2)V \wedge J + 2Hi(\sin^2\theta e^{2i\gamma}\Omega - \cos^2\theta\overline{\Omega}), \tag{B.35}$$

$$X^3 = -H^{-1}g^3 V \wedge J - H\sin 2\theta(e^{i\gamma}\Omega + c.c.). \tag{B.36}$$

C. Necessary and sufficient conditions for supersymmetry

The type of structure defined by the Killing spinor is determined by its intrinsic torsion. We have an $SU(3)$ structure in seven dimensions, specified by (V, J, Ω), with an associated six dimensional structure, specified by (J, Ω), or equivalently by the chiral unit norm spinor η. Given such a six dimensional structure, there is no obstruction to finding a connection ∇' that preserves it, $\nabla'\eta = 0$. ∇' is not unique, and different inequivalent classes of structure preserving connections are parametrized by the part of the torsion tensor called the intrinsic torsion. In [31] it is shown that for an $SU(3)$ structure in 6 dimensions there are 5 modules of the intrinsic torsion given by

$$d\Omega^{(2,2)} = \mathcal{W}_1 J \wedge J + \mathcal{W}_2 J, \tag{C.1}$$

$$\mathcal{W}_3 = (d J)^{(2,1)_0}, \tag{C.2}$$

$$\mathcal{W}_4 = \frac{1}{2} J \lrcorner dJ, \tag{C.3}$$

$$\mathcal{W}_5 = \frac{1}{2} Re\,\Omega \lrcorner d(Re\,\Omega). \tag{C.4}$$

The intrinsic torsion may be calculated in terms of the metric and matter fields by applying the Killing spinor equation to the spinor bilinears. Furthermore, the vanising of $\delta\lambda$ relates various components of the bosonic fields to each other. In this appendix we will determine the constraints on the bosonic fields of the minimal and gauged theories implied by the existence a timelike Killing spinor. As we shall see, most of the field content of the theories is determined by the structure. As a final step, we will show that the constraints we derive on the bosonic fields are also sufficient to ensure the existence of a timelike Killing spinor, and thus that we have derived the most general bosonic field configuration compatible with timelike supersymmetry. We will first work out the constrints for the minimal theory, and the results are then straightforwardly modified to account for the gauging.

C.1 Differential constraints

The various bispinors satisfy differential equations as a consequence of the Killing spinor equation. Using

$$\nabla_\mu(\epsilon^{aT} A \epsilon^b) = (\nabla_\mu \epsilon^a)^T A \epsilon^b + \epsilon^{aT} A(\nabla_\mu \epsilon^b) \tag{C.5}$$

where A is any matrix in the Clifford algebra, and employing the Killing spinor equation in the minimal theory, we may deduce

$$dg^A = -\frac{1}{5}\epsilon^{ABC} F^B \lrcorner X^C - \frac{8}{5} i_V F^A - \frac{2}{5} X^A \lrcorner G, \tag{C.6}$$

$$\nabla_\mu V_\nu = \frac{1}{5}\left(3G \lrcorner \star V + 2HJ \lrcorner G + \frac{1}{2} F^A \lrcorner \star X^A - 2g^A F^A\right)_{\mu\nu}, \tag{C.7}$$

$$d(HJ)_{\mu\nu\sigma} = \frac{1}{5}\left(3F^A_{\alpha[\mu} X^A{}_{\nu\sigma]}{}^\alpha - 2HG_{\alpha\beta\gamma[\mu} \star J_{\nu\sigma]}{}^{\alpha\beta\gamma} + 6i_V G_{\mu\nu\sigma}\right), \tag{C.8}$$

$$dX^A_{\mu\nu\sigma\tau} = \frac{1}{5}\Big(8F^A \lrcorner \star V_{\mu\nu\sigma\tau} + 12HF^A \wedge J_{\mu\nu\sigma\tau} + 4\epsilon^{ABC} F^B_{\alpha[\mu} \star X^C_{\nu\sigma\tau]}{}^\alpha$$
$$- 8g^A G_{\mu\nu\sigma\tau} + 6G_{[\mu\nu}{}^{\alpha\beta} \star X^A_{\sigma\tau]\alpha\beta}\Big). \tag{C.9}$$

where $i_V A$ denotes the vector V contracted on the first index of the form A. Note that (C.7) implies that V is Killing. Next, by successively contracting $\delta\lambda = 0$ with ϵ^{aT}, $\epsilon^{aT}\Gamma_\mu$,..., $\epsilon^{aT}\Gamma_{\mu\nu\sigma}$, and splitting the symplectic Majorana indices into symmetric and and antisymmetric parts, we find (among others) the following constraints:

$$g^A \partial_\mu \phi = = \frac{1}{5}\left(-2i_V F^A_\mu + 2X^A \lrcorner G_\mu + \epsilon^{ABC} F^B_{\alpha\beta} \lrcorner X^{C\alpha\beta}_\mu\right), \tag{C.10}$$

$$(d\phi \wedge V)_{\mu\nu} = \frac{1}{5}\left(2G \lrcorner \star V - 2HJ \lrcorner G - \frac{1}{2} F^A \lrcorner \star X^A - \frac{1}{2} g^A F^A\right)_{\mu\nu}, \tag{C.11}$$

$$2H(d\phi \wedge J)_{\mu\nu\sigma} = \frac{1}{5}\left(3F^A_{\alpha[\mu} X^A{}_{\nu\sigma]}{}^\alpha - 2HG_{\alpha\beta\gamma[\mu} \star J_{\nu\sigma]}{}^{\alpha\beta\gamma} - 4i_V G_{\mu\nu\sigma}\right), \tag{C.12}$$

$$(d\phi \wedge X^A)_{\mu\nu\sigma\tau} = \frac{1}{5}\Big(-2F^A \lrcorner \star V_{\mu\nu\sigma\tau} + 2HF^A \wedge J_{\mu\nu\sigma\tau} + 4\epsilon^{ABC} F^B_{\alpha[\mu} \star X^C_{\nu\sigma\tau]}{}^\alpha$$
$$+ 2g^A G_{\mu\nu\sigma\tau} + 6G_{[\mu\nu}{}^{\alpha\beta} \star X^A_{\sigma\tau]\alpha\beta}\Big). \tag{C.13}$$

Combining (C.6) with (C.10) we obtain

$$i_V F^A = -\frac{e^{-\phi}}{2} d(g^A e^\phi). \tag{C.14}$$

Given the Bianchi identity for F, and, as we will show below, that $\mathcal{L}_V \phi = 0$, this implies that

$$\mathcal{L}_V F^a = \mathcal{L}_V g^A = 0. \tag{C.15}$$

Next (C.7) and (C.11) combine to give

$$e^{-2\phi} d(e^{2\phi} V) = 2i_V \star G - g^A F^A. \tag{C.16}$$

We also find

$$\frac{e^{2\phi}}{2}d(He^{-2\phi}J) = i_V G \tag{C.17}$$

which (given the Bianchi identity for G, and $\mathcal{L}_V H = 0$) implies that

$$\mathcal{L}_V G = \mathcal{L}_V J = 0. \tag{C.18}$$

Finally, (C.9) and (C.13) give

$$\frac{e^{\phi}}{2}d(e^{-\phi}X^A) = F^A \lrcorner \star V + HF^A \wedge J - g^A G. \tag{C.19}$$

Given the form (B.35), (B.36) of the X^A, we may deduce that

$$\mathcal{L}_V \Omega = 0. \tag{C.20}$$

Therefore V generates a symmetry not just of the metric and matter fields but also of the G-structure.

C.2 Constraints from $\delta\lambda^a = 0$

In order to deduce the constraints on the bosonic fields of the theory implied by the vanishing of $\delta\lambda^a$, rather than solving (C.10)-(C.13) directly, it is technically much more convenient to break the manifest global $SU(2)$ symmetry of the theory, and to work with the Dirac spinor η. To this end, let us define

$$\tilde{F}^a{}_b = (S^{-1}FS)^a{}_b, \tag{C.21}$$

$$K = \tilde{F}^3, \tag{C.22}$$

$$L = \tilde{F}^1 - i\tilde{F}^2. \tag{C.23}$$

Then $\delta\lambda^a = 0$ implies the following pair of equations:

$$(\frac{5}{2}\partial_\mu\phi\Gamma^\mu + \frac{i}{4}K_{\mu\nu}\Gamma^{\mu\nu} - \frac{1}{6}\star G_{\mu\nu\sigma}\Gamma^{\mu\nu\sigma})\eta + \frac{1}{4}L_{\mu\nu}\Gamma^{\mu\nu}\eta^\star = 0, \tag{C.24}$$

$$(\frac{5}{2}\partial_\mu\phi\Gamma^\mu - \frac{i}{4}K_{\mu\nu}\Gamma^{\mu\nu} - \frac{1}{6}\star G_{\mu\nu\sigma}\Gamma^{\mu\nu\sigma})\eta^\star + \frac{1}{4}\overline{L}_{\mu\nu}\Gamma^{\mu\nu}\eta = 0. \tag{C.25}$$

Taking the complex conjugate of the second equation, and employing the reality properties of the gamma matrices together with the projection $\Gamma_0\eta = i\eta$, we may rewrite these equations in the following form:

$$(-5\partial_0\phi + \frac{1}{2}K_{ij}\Gamma^{ij} + \star G_{0ij}\Gamma^{ij})\eta - L_{0i}\Gamma^i\eta^\star = 0, \tag{C.26}$$

$$(5\partial_i\phi\Gamma^i - K_{0i}\Gamma^i - \frac{1}{3}\star G_{ijk}\Gamma^{ijk})\eta + \frac{1}{2}L_{ij}\Gamma^{ij}\eta^\star = 0. \tag{C.27}$$

What we have done is to split the supersymmetry variation of $\delta\lambda$ into positive and negative chirality parts on the base. To solve these equations we decompose the

above forms into $SU(3)$ irreducible representations. Successively contracting with η^T, $\overline{\eta}$, $\eta^T \Gamma_i$,..., $\overline{\eta} \Gamma^{ijk}$ we deduce that $\delta \lambda^a = 0$ is equivalent to

$$\partial_0 \phi = 0, \tag{C.28}$$

$$(\frac{1}{2} K_{ij} + \star G_{0ij})^{(2,0)+(0,2)+(0,0)} = -\frac{1}{8}(L_{0k}\Omega^k{}_{ij} + c.c.), \tag{C.29}$$

$$\frac{1}{3}\Omega_{ijk} \star G^{ijk} = \frac{i}{2}\overline{L}_{ij}J^{ij}, \tag{C.30}$$

$$\frac{1}{2}(\delta_i^j - iJ_i{}^j)(5\partial_j\phi - K_{0j} + i \star G_{jkl}J^{kl}) = -\frac{1}{4}L_{jk}\Omega_i{}^{jk}. \tag{C.31}$$

The primitive forms $\star G^{(2,1)_0+(1,2)_0}_{ijk}$, $\star G^{(1,1)_0}_{0ij}$, $K^{(1,1)_0}_{ij}$ and $L^{(1,1)_0}_{ij}$, traceless with respect to J, drop out of the supersymmetry variation and are unconstrained here.

Now we will re-express the differential constraints on the structure derived in the last subsection in terms of the transformed two forms. Let us define the rotated quantities

$$\tilde{g}^a{}_b = (S^{-1}gS)^a{}_b, \tag{C.32}$$

$$\tilde{X}^a{}_b = (S^{-1}XS)^a{}_b. \tag{C.33}$$

They take the form

$$\tilde{g}^1 - i\tilde{g}^2 = 0, \tag{C.34}$$

$$\tilde{g}^3 = -2H, \tag{C.35}$$

$$\tilde{X}^1 - i\tilde{X}^2 = -2iH\overline{\Omega}, \tag{C.36}$$

$$\tilde{X}^3 = 2V \wedge J. \tag{C.37}$$

Also define

$$C_\mu{}^a{}_b = i(S^{-1})^a{}_c \partial_\mu S^c{}_b. \tag{C.38}$$

Then we find

$$K_{0\mu} = \frac{e^{-\phi}}{H}\partial_\mu(He^\phi), \tag{C.39}$$

$$L_{0\mu} = i(C^1 - iC^2), \tag{C.40}$$

$$\frac{e^{-2\phi}}{2H}d(e^{2\phi}V)_{ij} = \star G_{0ij} + K_{ij}, \tag{C.41}$$

$$\frac{e^{2\phi}}{2H}d(He^{-2\phi}J)_{ijk} = G_{0ijk}, \tag{C.42}$$

$$i \star_6 (\frac{e^\Phi}{H}d(e^{-\Phi}H\Omega) + iC^3 \wedge \Omega)_{ij} = \overline{L}^{(0,0)}_{ij} - 2\overline{L}^{(1,1)_0}_{ij}, \tag{C.43}$$

where $\star_6$ denotes the Hodge dual on the base. Now, we can calculate all modules of the intrinsic torsion of the $SU(3)$ structure on the base using (C.42), (C.43), obtaining the equations quoted in (3.13)-(3.17). These, together with equations (C.28)-(C.31) and (C.39)-(C.43) (which are rewritten as in (3.9)-(3.12)) are necessary conditions for supersymmetry. In fact they are also sufficient, as we now show.

C.3 Sufficient conditions for supersymmetry in the minimal theory

The necessary conditions we have obtained gaurantee the vanishing of $\delta\lambda$, as may be seen by substituting (C.28)-(C.31) back in and employing the projections satisfied by η. It remains to be verified that a solution of the Killing spinor equation always exists. Rewriting the Killing spinor equation in terms of η we get

$$D_\mu \eta^a + \frac{1}{2}\partial_\mu(\log H)\eta^a + C_\mu^A(T^A)^a{}_b\eta^b - \frac{i}{10}F_{\lambda_1\lambda_2}{}^a{}_b\left(\Gamma_\mu{}^{\lambda_1\lambda_2} - 8\delta_\mu^{l_1}\Gamma^{\lambda_2}\right)\eta^a$$
$$+ \frac{1}{80}G_{\lambda_1\ldots\lambda_4}\left(\Gamma_\mu{}^{\lambda_1\ldots\lambda_4} - \frac{8}{3}\delta_\mu^{\lambda_1}\Gamma^{\lambda_2\lambda_3\lambda_4}\right)\eta^a = 0. \tag{C.44}$$

These two equations are not independent as can be seen by taking the complex conjugation of the second one. Separating the negative and the positive chirality parts we get two different equations: one is algebraic and the other differential. The first one reads

$$0 = i(C^1 + iC^2)\eta^* - \frac{1}{4}\omega_{\mu\lambda_1\lambda_2}(\Gamma - \Gamma^*)^{\lambda_1\lambda_2}\eta$$
$$+ \frac{i}{20}\left\{K_{\lambda_1\lambda_2}\left[(\Gamma - \Gamma^*)_\mu{}^{\lambda_1\lambda_2} - 8\delta_\mu^{l_1}(\Gamma - \Gamma^*)^{\lambda_2}\right]\eta - i\overline{L}_{\lambda_1\lambda_2}\left[(\Gamma + \Gamma^*)_\mu{}^{\lambda_1\lambda_2}\right.\right.$$
$$\left.\left. - 8\delta_\mu^{l_1}(\Gamma + \Gamma^*)^{\lambda_2}\right]\eta^*\right\} - \frac{1}{80}G_{\lambda_1\ldots\lambda_4}\left[(\Gamma - \Gamma^*)_\mu{}^{\lambda_1\ldots\lambda_4} - \frac{8}{3}\delta_\mu^{\lambda_1}(\Gamma - \Gamma^*)^{\lambda_2\lambda_3\lambda_4}\right]\eta. \tag{C.45}$$

Nothing that (C.16) implies the following expressions for the spin connection,

$$\omega_{0i0} = K_{0i} - \partial_i\phi, \tag{C.46}$$
$$\omega_{ij0} = \omega_{0ij} = \star G_{0ij} + K_{ij}, \tag{C.47}$$

one may verify that the algebraic equation (C.45) is satisfied. Next, the remaining differential equation for η is given by

$$(\partial_\mu + \frac{1}{4}\omega_{\mu ij}\Gamma^{ij})\eta + \partial_\mu(\log H)\eta + \frac{1}{2}C_\mu^3\eta - \frac{i}{40}\left\{K_{\lambda_1\lambda_2}\left[(\Gamma + \Gamma^*)_\mu{}^{\lambda_1\lambda_2}\right.\right.$$
$$\left.\left. - 8\delta_\mu^{l_1}(\Gamma + \Gamma^*)^{\lambda_2}\right]\eta - i\overline{L}_{\lambda_1\lambda_2}\left[(\Gamma - \Gamma^*)_\mu{}^{\lambda_1\lambda_2} - 8\delta_\mu^{l_1}(\Gamma - \Gamma^*)^{\lambda_2}\right]\eta^*\right\}$$
$$+ \frac{1}{160}G_{\lambda_1\ldots\lambda_4}\left[(\Gamma + \Gamma^*)_\mu{}^{\lambda_1\ldots\lambda_4} - \frac{8}{3}\delta_\mu^{\lambda_1}(\Gamma + \Gamma^*)^{\lambda_2\lambda_3\lambda_4}\right]\eta = 0. \tag{C.48}$$

The $\mu = 0$ component reduces to

$$\partial_0\eta = 0, \tag{C.49}$$

and is satisfied by any time-independent spinor satisfying the required projections. Next, consider the $\mu = i$ components. A straightforward but long calculation shows that it can be rewritten as

$$\left[\nabla_m + \frac{1}{4}\mathcal{W}_4^a\,\Gamma_{am} + \frac{i}{4}(3\mathcal{W}_4 + 2\mathcal{W}_5)^r\,J_{rm} - \frac{1}{32}\mathcal{W}_1^*\,\Omega_{ma_1a_2}\Gamma^{a_1a_2} - \frac{i}{8}\mathcal{W}_{3ma_1a_2}\Gamma^{a_1a_2}\right.$$
$$\left. + \frac{i}{32}\mathcal{W}_{2mj}^*\,\Omega^j{}_{a_1a_2}\Gamma^{a_1a_2}\right]\eta = 0 \tag{C.50}$$

As we show in appendix D, this is the most general $SU(3)$ preserving connection on the base. Therefore we always have a solution of the Killing spinor equation, and we have derived necessary and sufficient conditions for supersymmetry.

C.4 Necessary and sufficient conditions for supersymmetry in the gauged theory

Having determined the constraints for supersymmetry in the minimal theory, it is a straightforward matter to modify the results to account for the gauging. We will therefore only briefly quote our results. However there is one point which deserves mention. A desirable feature we want to maintain in the gauged theory is that the Killing vector V generates a symmetry not just of the metric and matter fields but also of the G-structure. The differential constraints on the structure in the gauged theory are

$$i_V F^A = -\frac{1}{2} e^{-\phi}(d(e^\phi g^A) + g\epsilon^{ABC} A^B e^\phi g^C), \qquad (C.51)$$

$$e^{-2\phi}d(e^{2\phi}V) = 2i_V \star G - g^A F^A - 8Hh e^{-4\phi}J, \qquad (C.52)$$

$$e^{2\phi}d(He^{-2\phi}J) = 2i_V G, \qquad (C.53)$$

$$e^\phi d(e^{-\phi}X^A) + g\epsilon^{ABC}A^B \wedge X^C = 2F^A \lrcorner \star V + 2HF^A \wedge J - 2g^A G + g \star X^A, \quad (C.54)$$

and V is again Killing. As before, from $\delta\lambda = 0$, we have $i_V d\phi = 0$. Therefore, imposing the gauge

$$i_V A^A = \frac{1}{2} e^\phi g^A, \qquad (C.55)$$

ensures that (given the Bianchi identities) the matter fields and the structure are also preserved along V. However, (C.55) does not entirely fix the gauge freedom; we may still perform time independent gauge transformations, under which both A_0^A and g^A transform in the adjoint. We may thus eliminate the scalars θ, γ altogether by imposing the gauge

$$A_0^1 = A_0^2 = 0, \quad A_0^3 = -e^\phi, \qquad (C.56)$$

and the structure simplifies to

$$g^1 = g^2 = 0, \qquad (C.57)$$

$$g^3 = -2H, \qquad (C.58)$$

$$X^1 - iX^2 = -2iH\overline{\Omega}, \qquad (C.59)$$

$$X^3 = 2V \wedge J. \qquad (C.60)$$

It is now a simple matter to modify the results of the minimal theory. What we find in the gauge (C.55), (C.56) is reported in eqs.(3.18)-(3.22).

D. Intrinsic torsion

In this appendix we derive a formula for the covariant derivative that leaves invariant the $SU(3)$ structure. The procedure to do it is well understood and we follow the lines of [22].

Write the connection as $\Gamma^i{}_{jk} = C^i{}_{jk} + K^i{}_{jk}$, where $C^i{}_{jk}$ are the Christoffel symbols and $K_{ijk} = K_{[i|j|k]}$ is the contorsion tensor. The contorsion is equivalent to the torsion $T^i{}_{jk}$ in that it satisfies

$$T^i{}_{jk} = 2K^i{}_{[jk]}, \tag{D.1}$$

$$K^i{}_{jk} = \frac{1}{2}\left(T^i{}_{jk} + T_j{}^i{}_k + T_k{}^i{}_j\right). \tag{D.2}$$

The contorsion is a tensor in $T^* \otimes so(6) \simeq (T^* \otimes su(3)) \oplus (T^* \otimes su(3)^\perp)$. $T^* \otimes su(3))$ decomposes under $SU(3)$ as

$$(\mathbf{3} + \overline{\mathbf{3}}) \times \mathbf{8} = (\mathbf{15} + \overline{\mathbf{15}}) + (\mathbf{6} + \overline{\mathbf{6}}) + (\mathbf{3} + \overline{\mathbf{3}}), \tag{D.3}$$

while $T^* \otimes su(3)^\perp$ as

$$(\mathbf{3} + \overline{\mathbf{3}}) \times (\mathbf{3} + \overline{\mathbf{3}} + \mathbf{1}) = (\mathbf{8} + \mathbf{8}') + (\mathbf{6} + \overline{\mathbf{6}}) + (\mathbf{3} + \overline{\mathbf{3}}) + (\mathbf{3}' + \overline{\mathbf{3}}') + (\mathbf{1} + \mathbf{1}'). \tag{D.4}$$

When acting on $SU(5)$ invariants, only this latter part of the contorsion contributes. Now we rewrite a general contorsion tensor according to its $SU(3)$ decomposition as

$$K_{lmn} = \left(L^{(1)}_m J_{ln} + L^{(2)}_{[l} J_{n]m} + L^{(3)}_{[l} g_{n]m}\right) + (k\Omega_{lmn} + c.c.) + (f_{mj}\Omega^j{}_{ln} + c.c)$$
$$+ (T^{(1)}_{mk}\overline{\Omega}^k{}_{ln} + T^{(2)}_{[l|j|}\overline{\Omega}^j{}_{n]m} + c.c.) + (U_{[ln]m} + c.c.), \tag{D.5}$$

where $K \in \mathbb{C}$ and, in complex notation, $f_{lm} = f_{\lambda\overline{\mu}} + f_{\overline{\lambda}\mu} \in \mathbb{R}$, $f_{\lambda\overline{\mu}}J^{\lambda\overline{\mu}} = 0$, $T_{lm} = T_{(\lambda\mu)}$, $U_{lmn} = U_{\overline{\lambda}(\mu\nu)}$, $U_{\overline{\lambda}\mu\nu}J^{\overline{\lambda}\mu} = 0$. Suppose that K_{lmn} is such $\nabla'J = 0 = \nabla'\Omega$, then the exterior derivative of J, Ω is given by

$$\frac{1}{6}dJ_{i_1i_2i_3} = K^r{}_{[i_1i_2}J_{|r|i_3]}, \tag{D.6}$$

$$\frac{1}{12}d\Omega_{i_1...i_4} = K^r{}_{[i_1i_2}\Omega_{|r|i_3i_4]}. \tag{D.7}$$

From these we calculate the intrinsic torsion modules and find

$$\mathcal{W}_1 = 8k^*, \tag{D.8}$$

$$\mathcal{W}_2 = 8if^*, \tag{D.9}$$

$$\mathcal{W}_{3i_1i_2i_3} = 6iT^{(1)*}_{[i_1|j|}\Omega^j{}_{i_2i_3]}, \tag{D.10}$$

$$\mathcal{W}_4 = L^{(3)} - L^{(2)} \lrcorner J, \tag{D.11}$$

$$\mathcal{W}_5 = -\frac{3}{2}L^{(3)} + 3\left(L^{(1)} + \frac{1}{6}L^{(2)}\right) \lrcorner J. \tag{D.12}$$

Some components of the contorsion are not determined by the $\mathcal{W}_i$ and are those under which the structure is preserved, corresponding to the freedom in choosing an $SU(3)$ preserving connection. As a matter of fact we can rewrite the contorsion now as

$$
\begin{aligned}
K_{lmn} = {} & \left(\frac{3}{2}\mathcal{W}_4 + \mathcal{W}_5\right)^r J_{r[l} J_{n]m} - \left(\frac{1}{2}\mathcal{W}_4 + \mathcal{W}_5\right)_{[l} g_{n]m} + \frac{1}{8}\left(\mathcal{W}_1^*\Omega_{lmn} + c.c.\right) \\
& - \left(\frac{i}{2}W_{3lmn}^* + c.c\right) + \left(\frac{i}{8}\mathcal{W}_{2mj}^* \Omega^j{}_{ln} + c.c.\right) \\
& + \left(L_m^{(1)} J_{ln} + 3L_{[l}^{(1)} J_{n]m} + 3L^{(1)r} J_{r[l} g_{n]m}\right) \\
& + \left[\left(T^{(2)} - 2T^{(1)}\right)_{[l|r|}\overline{\Omega}^r{}_{n]m} + c.c.\right] + \left(U_{[ln]m} + c.c.\right).
\end{aligned} \tag{D.13}
$$

One can directly check that the last two lines leave both J and Ω invariant or, equivalently, η. The first two lines therefore define the intrinsic contorsion K_{lmn}^0. Notice also that the combination $\frac{3}{2}\mathcal{W}_4 + \mathcal{W}_5$ is conformally invariant [31]. The last equation we need is the form assumed by $\nabla'\eta = 0$ which concretely reads

$$
\begin{aligned}
\Big[\nabla_m &+ \frac{1}{8}\left(3\mathcal{W}_4 + 2\mathcal{W}_5\right)^r J_{ra_1} J_{a_2m}\Gamma^{a_1a_2} - \frac{1}{8}\left(\mathcal{W}_4 + 2\mathcal{W}_5\right)_{a_1}\Gamma^{a_1}{}_m \\
& - \frac{1}{32}\left(\mathcal{W}_1^*\Omega_{ma_1a_2} + \mathcal{W}_1\overline{\Omega}_{ma_1a_2}\right)\Gamma^{a_1a_2} + \frac{i}{8}\left(\mathcal{W}_{3ma_1a_2}^* - \mathcal{W}_{3ma_1a_2}\right)\Gamma^{a_1a_2} \\
& + \frac{i}{32}\left(\mathcal{W}_{2mj}^* \Omega^j{}_{a_1a_2} - \mathcal{W}_{2mj}\overline{\Omega}^j{}_{a_1a_2}\right)\Gamma^{a_1a_2}\Big]\eta = 0.
\end{aligned} \tag{D.14}
$$

Using the projection (B.16) and the following we see that (D.14) can be rewritten as

$$
\begin{aligned}
\Big[\nabla_m &+ \frac{1}{4}\mathcal{W}_4^a\,\Gamma_{am} + \frac{i}{4}\left(3\mathcal{W}_4 + 2\mathcal{W}_5\right)^r J_{rm} - \frac{1}{32}\mathcal{W}_1^* \Omega_{ma_1a_2}\Gamma^{a_1a_2} - \frac{i}{8}\mathcal{W}_{3ma_1a_2}\Gamma^{a_1a_2} \\
& + \frac{i}{32}\mathcal{W}_{2mj}^* \Omega^j{}_{a_1a_2}\Gamma^{a_1a_2}\Big]\eta = 0
\end{aligned} \tag{D.15}
$$

E. Integrability Conditions

Let us define $\delta\lambda = \Delta_\lambda\epsilon$, $\delta\psi_\mu = \mathcal{D}_\mu\epsilon$. Then we may obtain the following integrability condition from commuting the Killing spinor equation with $\delta\lambda$:

$$
\begin{aligned}
\sqrt{5}\Gamma^\mu[\mathcal{D}_\mu, \Delta_\lambda]\epsilon^a = {} & \left(\frac{1}{2}P + \frac{1}{6}Q_{\mu\nu\sigma}\Gamma^{\mu\nu\sigma} + \frac{e^{2\phi}}{96}d(e^{-2\phi}G)_{\mu\nu\sigma\tau\rho}\Gamma^{\mu\nu\sigma\tau\rho}\right)\epsilon^a \\
& + \left(iR_\mu^A\Gamma^\mu + \frac{ie^{-\phi}}{6}d(e^\phi F^A)_{\mu\nu\sigma}\Gamma^{\mu\nu\sigma}\right)T^{Aa}{}_b\epsilon^b \\
& + \sqrt{5}\left(\frac{1}{60}G_{\mu\nu\sigma\tau}\Gamma^{\mu\nu\sigma\tau}\delta_b^a + \frac{3i}{5}F_{\mu\nu}^A\Gamma^{\mu\nu}T^{Aa}{}_b\right)\delta\lambda^b,
\end{aligned} \tag{E.1}
$$

where P, Q, R are defined by eqs.(2.7,2.8,2.9), and the dilaton, four form and two form field equations are respectively $P = 0$, $Q = 0$, $R^A = 0$. Imposing the dilaton and

four form field equations, and the Bianchi identities for the forms, the integrability condition reduces to

$$R^A_\mu \Gamma^\mu T^{Aa}{}_b \epsilon^b = 0. \tag{E.2}$$

As in the analysis of $\delta\lambda = 0$, it is convenient to transform this expression and work in terms of η. Writing $\tilde{R} = S^{-1}RS$, and using the same procedure as before, we may deduce

$$\tilde{R}^3_\mu = 0, \tag{E.3}$$

$$(\tilde{R}^1 - i\tilde{R}^2)_0 = 0, \tag{E.4}$$

$$(\delta^j_i + iJ_i{}^j)(\tilde{R}^1 - i\tilde{R}^2)_j = 0. \tag{E.5}$$

Next we consider the integrability condition for the Killing spinor equation. After a long calculation we obtain

$$
\begin{aligned}
\Gamma^\nu[\mathcal{D}_\mu, \mathcal{D}_\nu]\epsilon^a = & \left[-\frac{1}{2}E_{\mu\nu}\Gamma^\nu + e^{2\phi}d(e^{-2\phi}G)^{\nu\sigma\tau\rho\xi}\left(-\frac{1}{120}g_{\mu\nu}\Gamma_{\sigma\tau\rho\xi} + \frac{1}{200}\Gamma_{\mu\nu\sigma\tau\rho\xi} \right) \right. \\
& \left. + \frac{1}{10}Q^{\nu\sigma\tau}\left(\frac{1}{2}\Gamma_{\mu\nu\sigma\tau} - g_{\mu\nu}\Gamma_{\sigma\tau} \right) \right]\epsilon^a \\
& + \left[\frac{ie^{-\phi}}{5}d(e^\phi F^A)^{\nu\sigma\tau}\left(2g_{\mu\nu}\Gamma_{\sigma\tau} + \frac{1}{6}\Gamma_{\mu\nu\sigma\tau} \right) - \frac{i}{5}R^{A\nu}(-4g_{\mu\nu} + \Gamma_{\mu\nu}) \right]T^{Aa}{}_b\epsilon^b \\
& + \left[\partial_\mu\phi\delta^a_b - \frac{i}{25}F^{A\nu\sigma}(8g_{\mu\nu}\Gamma_\sigma - \Gamma_{\mu\nu\sigma})T^{Aa}{}_b \right. \\
& \left. + \frac{1}{25}G^{\nu\sigma\tau\rho}\left(-\frac{2}{3}g_{\mu\nu}\Gamma_{\sigma\tau\rho} + \frac{1}{4}\Gamma_{\mu\nu\sigma\tau\rho} \right)\delta^a_b \right]\delta\lambda^b = 0. \tag{E.6}
\end{aligned}
$$

Given the Bianchi identites and the field equations imposed for and implied by the vanishing of the integrability condition for $\delta\lambda$, and converting to the dirac spinor η, this reduces to the pair of equations

$$-\frac{1}{2}E_{\mu\nu}\Gamma^\nu\eta - \frac{1}{5}(\tilde{R}^1 - i\tilde{R}^2)^j(-4g_{\mu j} + \Gamma_{\mu j})\eta^\star = 0, \tag{E.7}$$

$$-\frac{1}{2}E_{\mu\nu}\Gamma^{\nu\star}\eta - \frac{1}{5}(\tilde{R}^1 - i\tilde{R}^2)^j(-4g_{\mu j} + \Gamma^\star_{\mu j})\eta^\star = 0. \tag{E.8}$$

Taking the i component we may deduce

$$E_{j0}\eta = 0, \tag{E.9}$$

$$-E_{ij}\Gamma^j\eta - \frac{2}{5}(\tilde{R}^1 - i\tilde{R}^2)_j(-4\delta^j_i + \Gamma_i{}^j)\eta^\star) = 0. \tag{E.10}$$

Hence $E_{j0} = 0$, while contracting the second expression with ϵ^T we obtain

$$(\tilde{R}^1 - i\tilde{R}^2)_j(-\delta^j_i + \frac{1}{5}(\delta^i_j + iJ_i{}^j)) = 0. \tag{E.11}$$

Thus $\tilde{R}^A_\mu = 0$, and so $R^A_\mu = 0$. Next contracting (E.10) with $\overline{\eta}\Gamma_k$ implies

$$E_{ij}(\delta^j_k \pm iJ_k{}^j) = 0, \tag{E.12}$$

so $E_{ij} = 0$, and the 0 component of (E.7) then implies that $E_{00} = 0$.

In summary, given the existence of a timelike Killing spinor, it is sufficient to impose the Bianchi identites and the four form and dilaton field equations. The remaining field equations are implied by supersymmetry.

In the gauged theory the structure of the integrability conditions is identical. The additional terms which arise are such that one now obtains the gauged theory field equations and Bianchi identities in precisely the same fashion, together with the additional terms $\sqrt{5}(8he^{-4\phi} - ge^{\phi})\delta\lambda$, $\frac{2}{\sqrt{5}}(m + 2he^{-4\phi})\delta\lambda$ in $\Gamma^{\nu}[\mathcal{D}_{\nu}, \Delta_{\lambda}]$, $\Gamma^{\nu}[\mathcal{D}_{\mu}, \mathcal{D}_{\nu}]$ respectively. Thus as in the ungauged theory it is sufficient to impose the Bianchi identities and the four form and dilaton field equations.

References

[1] J. M. Maldacena and C. Nunez, *Supergravity description of field theories on curved manifolds and a no go theorem*, Int. J. Mod. Phys. **A16** (2001) 822-855, `hep-th/0007018`.

[2] J. M. Maldacena and C. Nunez, *Towards the large N limit of pure N=1 super Yang Mills*, Phys. Rev. Lett. 86 (2001) 588-591, hep-th/0008001.

[3] J. P. Gauntlett, *Branes, calibrations and supergravity*, hep-th/0305074.

[4] D. J. Smith, *Intersecting brane solutions in string and M-theory*, Class. Quant. Grav. **20** (2003) R233, `hep-th/0210157`.

[5] B. S. Acharya, J. M. Figueroa-O'Farrill, C. M. Hull, B. Spence, *Branes at conical singularities and holography*, Adv. Theor. Math. Phys. **2** (1999) 1249-1286, `hep-th/9808014`.

[6] I. R. Klebanov, M. J. Strassler, *Supergravity and a Confining Gauge Theory: Duality Cascades and χSB-Resolution of Naked Singularities*, JHEP **0008** (2000) 052, `hep-th/0007191`.

[7] C. Nunez, I. Y. Park, M. Schvellinger, T. A. Tran, *Supergravity duals of gauge theories from $F(4)$ gauged supergravity in six dimensions*, JHEP **0104** (2001) 025, `hep-th/0103080`.

[8] K. Tod, *All metrics admitting supercovariantly constant spinors*, Phys. Lett. **B121** (1983) 241-244.

[9] J. Gauntlett, J. Gutowski, C. Hull, S. Pakis and H. Reall, *All supersymmetric solutions of minimal supergravity in five dimensions*, Class. Quant. Grav. **20** (2003) 4587.

[10] J. P. Gauntlett, D. Martelli, S. Pakis, and D. Waldram, *G-Structures and Wrapped NS5-Branes*, Commun. Math. Phys. **247** (2004) 421-445, `hep-th/0205050`.

[11] D. Martelli and J. Sparks, *G-Structures, fluxes and calibrations in M-theory*, Phys. Rev. **D68** (2003), 085014, `hep-th/0306225`.

[12] J. Gutowski, D. Martelli and H. Reall, *All supersymmetric solutions of minimal supergravity in six dimensions*, Class. Quant. Grav. **20** (2003) 5049-5078, `hep-th/0306235`.

[13] J. P. Gauntlett, D. Martelli, D. Waldram, *Superstrings with Intrinsic Torsion*, Phys.Rev. **D69** (2004) 086002, `hep-th/0302158`.

[14] J. P. Gauntlett and J. B. Gutowski, *All supersymmetric solutions of minimal gauged supergravity in five dimensions*, Phys. Rev. **D68** (2003) 105009, `hep-th/0304064`.

[15] M. Cariglia and O. A. P. Mac Conamhna, *The general form of supersymmetric solutions of N=(1,0) U(1) and SU(2) gauged supergravities in six dimensions*, Class. Quant. Grav. **21** (2004) 3171 - 3195, `hep-th/0402055`.

[16] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, *Supersymmetric AdS$_5$ solutions of M-theory*, `hep-th/0402153`.

[17] A. Lukas and P.M. Saffin, *M-theory compactification, fluxes and AdS$_4$*, `hep-th/0403235`.

[18] G. Dall'Agata, *On supersymmetric solutions of type IIB supergravity with general fluxes*, `hep-th/0403220`.

[19] E. J. Hackett-Jones and D. J. Smith, *Type IIB Killing spinors and calibrations*, `hep-th/0405098`.

[20] A. R. Frey, *Notes on SU(3) Structures in Type IIB Supergravity*, `hep-th/0404107`.

[21] S. L. Cacciatori, M. M. Caldarelli, D. Klemm, D. S. Mansi, *More on BPS solutions of N=2, D=4 gauged supergravity*, `hep-th/0406238`.

[22] J. P. Gauntlett and S. Pakis, *The Geometry of $D = 11$ Killing Spinors*, JHEP **0304** (2003) 039, `hep-th/0212008`.

[23] J. Gauntlett, J. Gutowski and S. Pakis, *The geometry of $D = 11$ null Killing spinors*, JHEP **0312** (2003) 049, `hep-th/0311112`.

[24] A. Salam and E. Sezgin, "Supergravities In Diverse Dimensions. Vol. 1, 2." SPIRES entry.

[25] P. K. Townsend and P. van Nieuwenhuizen, *Gauged Seven-Dimensional Supergravity*, Phys. Lett. **B125** (1983) 41.

[26] L. Mezincescu, P. K. Townsend and P. van Nieuwenhuizen, *Stability Of Gauged $D = 7$ Supergravity And The Definition Of Masslessness In (Ads) In Seven-Dimensions*, Phys. Lett. **B143** (1984) 384.

[27] P. K. Townsend, *String-Membrane Duality in Seven Dimensions*, Phys. Lett. **B354** (1995) 247-255, `hep-th/9504095`.

[28] A. H. Chamseddine and W.A. Sabra, *$D = 7$ SU(2) gauged supergravity from D=10 supergravity*, Phys. Lett. **B476** (2000) 415-419, `hep-th/9911180`.

[29] H. Lu and C.N. Pope, *Exact embedding of N=1, D=7 gauged supergravity in D=11*, Phys. Lett. **B467** (1999) 67-72, `hep-th/9906168`.

[30] P. K. Townsend, K. Pilch and P. Van Nieuwenhuizen, *Self-duality in odd dimensions*, Phys. Lett **B136** (1984) 38, Addendum-ibid **B137** (1984) 443.

[31] S. Chiossi and S. Salamon, *The intrinsic torsion of $SU(3)$ and G_2 structures*, Differential Geometry, Valencia 2001, World Sci. Publishing, 2002, pp 115-133, `math.DG/0202282`.

[32] G. W. Gibbons and P. Rychenkova, *Single-sided domain walls in M-theory*, J.Geom.Phys. **32** (2000) 311-340, `hep-th/9811045`.

[33] R. Britto-Pacumio, A. Strominger and A. Volovich, *Holography for Coset Spaces*, JHEP **9911** (1999) 013, `hep-th/9905211`.

[34] J. P. Gauntlett, N. Kim and D. Waldram, *M-Fivebranes Wrapped on Supersymmetric Cycles*, Phys.Rev. D63 (2001) 126001, hep-th/0012195.